Pergamon

PII: S0021-8928(96)00053-6

J. Appl. Maths Mechs, Vol. 60, No. 3, pp. 421–427, 1996

0021-8928/96 $24.00+0.00

ANALYTIC SOLUTIONS FOR INTERMEDIATE-THRUST ARCS OF ROCKET TRAJECTORIES IN A NEWTONIAN FIELD†

D. M. AZIMOV

Tashkent

(*Received* 21 *December* 1993)

Mayer's variational problem of determining the optimum trajectories of a rocket moving with constant exhaust velocity and bounded mass flow rate in a Newtonian field is considered. New analytic solutions are obtained for plane intermediate-thrust arcs, using the canonical system of equations of the variational problem and the properties of the switching function. These solutions represent certain spiral trajectories. In motion with a fixed time, at arbitrary angular distances, these solutions satisfy Robbins' necessary optimum condition. As an example the problem of minimizing the characteristic velocity of flight between elliptic orbits is considered. Copyright © 1996 Elsevier Science Ltd.

1. As we know, the optimum trajectory of a rocket in a Newtonian field may consist of arcs of zero thrust (ZT), intermediate thrust (IT) and maximum thrust (MT) [1]. In a spherical system of coordinates with origin at the attracting centre, the motion of a rocket along these arcs is governed by a system of canonical equations for Mayer's variational problem [2]

$$\dot{\mathbf{v}} = \frac{cm}{M}\frac{\boldsymbol{\lambda}}{\lambda} - \frac{\mu}{r^3}\mathbf{r}, \quad \dot{\mathbf{r}} = \mathbf{v}, \quad \dot{M} = -m$$

$$\dot{\boldsymbol{\lambda}} = \boldsymbol{\lambda}_r, \quad \dot{\boldsymbol{\lambda}}_r = \frac{\mu}{r^3}\boldsymbol{\lambda} - 3\frac{\mu}{r^5}(\boldsymbol{\lambda}\mathbf{r})\mathbf{r}, \quad \dot{\lambda}_7 = \frac{cm}{M^2}\lambda \tag{1.1}$$

with Hamiltonian

$$H = -(\boldsymbol{\lambda}\mathbf{r})\frac{\mu}{r^3} + (\boldsymbol{\lambda}_r\mathbf{v}) + \varkappa m$$

where $r = (r, 0, 0)$ is the radius vector with initial point at the attracting centre, $\mathbf{v} = (\upsilon_1, \upsilon_2, \upsilon_3)$ is the velocity, M is the mass, $c = \text{const}$ is the exhaust velocity, $m (0 \leqslant m \leqslant m)$ is the mass flow rate, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ is the basis-vector, $\boldsymbol{\lambda}_r = (\lambda_4, \lambda_5, \lambda_6)$ is the vector conjugate to the radius vector, $\boldsymbol{\lambda}_7$ is a factor conjugate to the mass, $\varkappa$ is the switching function. The unit vector $\mathbf{1} = \boldsymbol{\lambda}/\lambda$, whose direction coincides with that of the thrust vector, and the quantity m are treated as control variables. The components of all vectors are given in spherical coordinates.

We know that the appearance of IT arcs is a degenerate case of the variational problem [3]. Despite the fact that, up to the present, various analytical solutions are known for IT arcs—a spiral [1, 4], circular [5] and spherical trajectories [6]‡—the existence of other solutions for IT arcs and their optimality are still unsolved problems. Here we shall show that in two dimensions the first integrals of the canonical system of equations of the variational problem

$$H = \lambda_1\left(\frac{\upsilon_2^2}{r} - \frac{\mu}{r^2}\right) - \lambda_2\frac{\upsilon_1\upsilon_2}{r} + \lambda_4\upsilon_1 + \lambda_5\frac{\upsilon_2}{r} = C$$

†*Prikl. Mat. Mekh.* Vol. 60, No. 3, pp. 426–432, 1996.

‡See also AZIMOV D. M., Investigation of optimum trajectories in a central Newtonian field. Candidate dissertation, Moscow, 1991.

$$\lambda_1 \nu_1 + \lambda_2 \nu_2 - 2\lambda_4 r + c\lambda \ln\left(\frac{M_0}{M}\right) - 3Ct = C_1 \tag{1.2}$$

$$\lambda_7 M = c\lambda = C_2, \quad \lambda_5 = C_3$$

where C, C_1, C_2 and C_3 are constants of integration, and the invariant relations

$$\lambda_1\lambda_4 + \lambda_1\lambda_2 \frac{\nu_2}{r} - \lambda_2^2 \frac{\nu_1}{r} + \lambda_5 \frac{\lambda_2}{r} = 0 \tag{1.3}$$

$$\lambda_4^2 + \left(\lambda_1 \frac{\nu_2}{r} - \lambda_2 \frac{\nu_1}{r} + \frac{\lambda_5}{r}\right)^2 = \lambda^2 \frac{\mu}{r^3} - 3\lambda_1^2 \frac{\mu}{r^3} \tag{1.4}$$

$$(\lambda^2 - 5\lambda_1^2)\, \nu_1 + 2\lambda_1(\lambda_1\nu_1 + \lambda_2\nu_2) - 4\lambda_1\lambda_4 r = 0 \tag{1.5}$$

that follow from the condition that the switching function vanish identically [5], enable us to obtain new analytic solutions of the canonical system of equations for IT arcs in a Newtonian field and to investigate them for optimality.

Thus, eliminating λ_4 from the first equation of (1.2) and the relation (1.4), using (1.3), we obtain the equations

$$(\lambda_1\nu_2 - \lambda_2\nu_1)^2 + \lambda_5(\lambda_1\nu_2 - \lambda_2\nu_1) = C\lambda_1 r + \lambda_1^2 \frac{\mu}{r}$$

$$(\lambda_1\nu_2 - \lambda_2\nu_1 + \lambda_5)^2 = \lambda_1^2 \frac{\mu}{r} - 3\mu \frac{\lambda_1^4}{r\lambda^2}$$

which yield an equation for r

$$C^2\lambda^4\lambda_1^2 r^4 + 6\mu C\lambda^2\lambda_1^5 r^2 + (3\mu\lambda^2\lambda_1^4\lambda_5^2 - \mu\lambda\lambda_1^2\lambda_5^2) r + 9\mu^2\lambda_1^8 = 0 \tag{1.6}$$

Depending on the sign of the discriminant

$$Q = \mu^4 \frac{\lambda_5^4}{C^8}\left(3\frac{\lambda_1^2}{\lambda^2} - 1\right)^2 \left[\frac{\lambda_5^4}{256}\left(3\frac{\lambda_1^2}{\lambda^2} - 1\right)^2 - \mu C \frac{\lambda_1^9}{\lambda^6}\right]$$

and the sign of the quantity $\lambda_1 C$, Eq. (1.6) can have the following solutions.

If $Q \geqslant 0$ and $\lambda_1 C < 0$, then

$$r = F_1 + \left(\frac{F}{F_1}\right)^{1/2} \tag{1.7}$$

where

$$F = \frac{\lambda_5^2\mu}{2C^2}(1 - 3s^2), \quad F_1 = \left[-\frac{2\mu\lambda s^3}{C}(1 + 2\operatorname{cosec}(2\alpha))\right]^{1/2}$$

$$\operatorname{tg}\alpha = \left(\operatorname{tg}\frac{\beta}{2}\right)^{1/3}, \quad |\alpha| \leqslant \frac{\pi}{4}, \quad 1 + 2\operatorname{cosec}(2\alpha) > 0$$

$$\sin\beta = \frac{16\lambda^3 s^9}{16\mu\lambda^3 s^9 - \lambda_5^4(3s^2 - 1)^2 / C}, \quad s = \sin\varphi$$

If

$$1 + 2\operatorname{cosec}(2\alpha) \leqslant 0$$

then the IT arcs have no real solutions.

If $Q < 0$ and $\lambda_1 C < 0$, then

$$r = F_2 + \left(\frac{F}{F_2}\right)^{1/2} \tag{1.8}$$

where

$$F_2 = \left[\frac{2\mu\lambda s^3}{C}\left(2\cos\frac{\alpha}{3} - 1\right)\right]^{1/2}, \quad 2\cos\frac{\alpha}{3} - 1 > 0$$

$$\cos\alpha = -\frac{16\mu\lambda^3 s^9 - \lambda s^4(1-3s^2)^2/(8C)}{16\mu\lambda^3 s^9}$$

If

$$2\cos\frac{\alpha}{3} - 1 \leqslant 0$$

then

$$r = F_3 + \left(\frac{F}{F_3}\right)^{1/2} \tag{1.9}$$

where

$$F_3 = \left[\frac{2\mu\lambda s^3}{C}\left(1 - 2\cos\left(\frac{\alpha}{3} + \frac{\pi}{3}\right)\right)\right]^{1/2}, \quad 1 - 2\cos\left(\frac{\alpha}{3} + \frac{\pi}{3}\right) > 0$$

If the last inequality does not hold, there are no real solutions for IT arcs. Suppose that, using one of formulae (1.7)–(1.9), we have determined the radius vector

$$r = R(\varphi) \tag{1.10}$$

where φ is the angle between the basis-vector and the perpendicular to the radius vector in Lawden's system of coordinates [1]. Then, using system (1.1), the last two equations of (1.2) and formulae (1.3)–(1.5), the other solutions of the canonical system may be expressed in terms of elementary functions and quadratures

$$\begin{gathered}
v_1 = \frac{2k}{\lambda}\gamma_2, \quad v_2 = \frac{1}{\lambda s}\gamma_1, \quad t = \frac{\lambda}{2}\int\frac{\rho}{k\gamma_2}d\varphi + C_4 \\
\theta = \frac{1}{2}\int\frac{\rho}{Rsk}\frac{\gamma_3}{\gamma_2}d\varphi + C_5, \quad M = M_0\exp\left(\frac{P}{C_2}\right) \\
\lambda_1 = \lambda s, \quad \lambda_2 = \lambda k, \quad \lambda_4 = \frac{(C_3 - \gamma_1)k}{Rs}, \quad \lambda_7 = \frac{C_2}{M} \\
\gamma_1 = \frac{1}{C_3}\left(3\mu\frac{\lambda^2 s^4}{R} + C\lambda s R + C_3^2\right), \quad \gamma_2 = \frac{3\gamma_1 + 2C_3}{5s^2 - 3}
\end{gathered} \tag{1.11}$$

$$\gamma_3 = \frac{(3-s^2)\gamma_1 + 6C_3k^2}{5s^2-3}, \quad k = \cos\varphi, \quad \rho = \frac{dr}{d\varphi}$$

$$P = C_1 - \frac{k}{s}\frac{\gamma_1(15s^2-3)+8C_3s^2}{5s^2-3} + 3C(t-C_4)$$

where C_4 and C_5 are constants of integration. If the quantities υ_1, υ_2, R, M, λ_4 have been determined, the equality $\varkappa^{(IV)} = 0$ yields the following expression for the mass flow rate

$$m = \frac{10\mu\lambda^2s^2 - \upsilon_2^2\lambda^2r(3-13s^2) - 2\lambda\lambda_4r^2(s\upsilon_1 - 3k\upsilon_2) - 4\lambda_4^2r^3 + 6\lambda sC_3\upsilon_2r^{-1}}{cs\lambda^2r^2(3-5s^2)} \tag{1.12}$$

Formulae (1.10)–(1.12) represent solutions of the canonical equations (1.1) for IT arcs without reference to the optimality criterion. It should be noted that when $\lambda_1 < 0$ and $C > 0$ the solutions (1.7) and (1.11) satisfy Robbins' necessary condition for optimality [3].

2. We now consider some special cases.

1. If the conditions of the variational problem do not imply that the time the rocket is in motion is fixed, and the functional of the problem is independent of time, then $C = 0$ [1]. Equation (1.6) then yields an expression for the radius vector

$$r = \frac{9\mu\lambda^2}{\lambda_5^2}\frac{s^6}{1-3s^2} \tag{2.1}$$

identical with the formula that defines Lawden's spiral [1] (see also Remark 1). Solutions of system (1.1) corresponding to these spirals have been obtained for the case $\lambda = 1$ [1, 7].

We will now present other solutions of system (1.1) for IT arcs, corresponding to (2.1). These solutions may be determined using the second and third equations of (1.2), Eqs (1.3) and (1.4), and the equations

$$\dot{r} = \upsilon_1, \quad \dot{\theta} = \upsilon_2 / r \tag{2.2}$$

in the form

$$\begin{aligned}
&\upsilon_1 = \frac{6C_3}{\lambda}\frac{1-3s^2}{s_1}, \quad \upsilon_2 = \frac{C_3}{3\lambda}\frac{1-3s^2}{s}\\
&t = \frac{9\mu\lambda^2}{C_3^2}\int\frac{s^5s_1}{(1-3s^2)^2}d\varphi + t_0, \quad \theta = \frac{1}{3}\int\frac{s_1-3s^4}{s^4}d\varphi + \theta_0\\
&M = \frac{M_0C}{\lambda C_2}\left[\frac{kC_3(-18s^6-39s^4+35s^2-6)}{3s^3s_1} - C_1\right].\\
&\lambda_1 = \lambda s, \quad \lambda_2 = \lambda k, \quad \lambda_4 = \frac{C_3^3(1-3s^2)^2}{27\mu\lambda^3s^9}\\
&\lambda_7 = \frac{C_2}{M}, \quad s_1 = 9s^4 - 17s^2 + 6
\end{aligned} \tag{2.3}$$

where t_0 and θ_0 are constants of integration. The quantity m is determined from (1.12), with r, υ_1, υ_2, M, λ_4 given by (2.1) and the appropriate formulae of system (2.3). The solutions (2.1) and (2.3) describe motion along spiral trajectories different from previously known spirals.

It can be shown that the second and third equations of system (1.2), Eqs (1.3)–(1.6) and the equations of system (2.2) also enable one to find two further classes of analytic solutions for system (1.1), but here the formulae for the radius vector are identical with (2.1). Since the fact that the Lawden spirals are no optimal was established using the formula for r [3], it follows that the solutions (2.1) and (2.3) presented above, as well as the two classes of analytic solutions whose derivation has just been described, will also not satisfy Robbins' necessary optimum condition.

Hence, by investigating the canonical system of equations, and taking the properties of the switching function into account, one can considerably extend the class of analytic solutions for plane IT arcs. Note that the solutions obtained in this section were obtained without reference to the objective functional.

Remark 1. It can be verified directly (using Lawden's equations of motion to investigate the IT arcs and the canonical system (1.1)) that the constant of integration A in the formulae for Lawden's spirals, the cyclic constant λ_5 and the magnitude of the radius vector satisfy the relation

$$\lambda_5 = -\lambda A \tag{2.4}$$

2. If the manoeuvre time is fixed ($C \neq 0$), the finite value of the polar angle is not given and the objective function is not an explicit function of the polar angle (this happens, in particular, in various problems associated with minimizing the characteristic velocity), then the cyclic integral (the last equality of system (1.2)) and the transversality conditions imply

$$\lambda_5 = 0 \tag{2.5}$$

over the entire optimum trajectory. We will derive analytic solutions for IT arcs in such cases.

When (2.5) is true, the first equation of (1.2) and relations (1.3) and (1.4) immediately give

$$r^2 = -3\mu\lambda C^{-1}s^3 \tag{2.6}$$

Here necessarily $s > 0, C < 0$ or $s < 0, C > 0$. The other solutions of system (1.1) corresponding to (2.5) and (2.6) may be found by using the second equation of system (1.2)–(1.4), (1.12), (2.2) in the form

$$\begin{gathered}
v_1 = \frac{6k\sqrt{z}}{\lambda(3-s^2)}, \quad v_2 = \frac{\sqrt{z}(9-7s^2)}{\lambda(3-s^2)}, \quad z = \lambda\left(Csr + \frac{\mu\lambda s^2}{r}\right) \\
t = \chi^3\left[4\mu^{1/2}\int\frac{s^{1/4}(3-s^2)}{(1+3s^2)}d\varphi\right]^{-1} + t_0 \\
\theta = \frac{9}{4}\chi\operatorname{ctg}\varphi - \frac{7}{4}\chi\varphi + \theta_0, \quad \chi = \left(\frac{3\mu\lambda}{C}\right)^{1/4} \\
M = M_0\exp\left[\frac{1}{C_2}\left(C_1 + 3Ct - \frac{k\sqrt{z}(3+s^2)}{\lambda s(3-s^2)}\right)\right]
\end{gathered} \tag{2.7}$$

where t_0 and θ_0 are constants of integration. It can be shown that the solutions (2.6) and (2.7) describe motion along certain spiral trajectories, also different from those already known.

Note that if the conditions $s < 0$ and $C > 0$ are satisfied, one can verify that the solution (2.6) satisfies Robbins' necessary optimum condition $\lambda sr < 0$ [3]. Consequently, the IT solutions (2.6) and (2.7) may be used in optimum flight problems with fixed time and arbitrary angular distance, and also when the objective functional does not depend explicitly on the angular distance in a Newtonian field [8]. An example will be presented below.

Remark 2. Since the solutions (2.1) and (2.3) contain the cyclic constant λ_5, while the solutions for Lawden's spirals involve the constant A, it would be interesting to consider these solutions taking the end conditions and transversality conditions into account. If there is no restriction on the angular distance, the cyclic constant satisfies condition (2.5). Consequently, the constant A will also vanish. In that case the IT arcs described by (2.1) and (2.3), as well as the Lawden spirals (see (2.4)), degenerate into ZT arcs. It follows that (2.1), (2.3) and the Lawden spirals can only be solutions for IT arcs if one is given the final value of the polar angle or if the objective functional depends explicitly on the polar angle. In particular, if one is considering the problem of minimizing the characteristic velocity [1, 7] and there is no restriction on the angular distance, the Lawden spirals are not solutions for IT arcs.

Example. Let us consider the problem of minimizing the characteristic velocity of flight between coplanar intersecting elliptic orbits in a central Newtonian field. At the starting time we have the following conditions: $l_1 = 0.8$; $p_1 = 8200$ km, $\omega_1 = -1.5$, and $c = 3$ km/s. The phase of motion along the initial orbit is assumed to be arbitrary. At the final instant of time one has the conditions $l_2 = 0.7, p_2 = 9500$ km, and $\omega_1 = 0.5$. The flight time is fixed in advance at $T = 500$ s. The phase of motion along the final orbit is also considered to be arbitrary. An impulse solution of this problem was obtained in [9].

We shall show that the flight just defined may be implemented with a single IT arc. In that case the trajectory will consist of two arcs: one with ZT and the other with IT, and the initial and final points of the latter are switching points. At these points one must ensure that the conditions of continuity hold for the basis vector, the radius vector and the velocity vector; the switching function must vanish there. Hence we have the following conditions for the first switching point

$$\begin{aligned}
&-3\mu\lambda\sin^3\varphi_1 = p_1^2 C/(1+e_1\cos f_1)^2 \\
&36\lambda_1\lambda_2^2\left(Cr^2+\frac{\lambda_1\mu}{r}\right) = e_1^2\frac{\mu}{p_1}\sin^2 f_1(3\lambda^2-\lambda_1^2)^2 \\
&(7\lambda_1^2-9\lambda^2)^2\left(\frac{Cr^2+\lambda_1\mu}{r}\right)\lambda_1 = \frac{\mu\lambda_1^2}{p_1}(1+e_1\cos f_1)(3\lambda^2-\lambda_1^2)^2 \\
&\lambda\sin\varphi_1 = B_1 e_1\sin f_1 + CI_2(f_1) \\
&\lambda\cos\varphi_1 = B_1(1+e_1\cos f_1)+\frac{D_1}{1+e_1\cos f_1}+CI_2(f_1) \\
&\frac{c}{M}\lambda-\lambda_7 = 0
\end{aligned} \tag{2.8}$$

At the second switching point we have conditions similar to (2.8) except that the subscript on the quantities φ, f, l, p, B and D, will now be 2. Here

$$I_2(f) = \frac{\operatorname{ctg} f}{l(1+e\cos f)}+\frac{1+e\cos f}{e\sin f}\sin f\int\frac{df}{\sin^2 f(1+e\cos f)^2}$$

In addition, at the final instant of time we have a transversality condition

$$\lambda_7 = -\partial J/\partial M_1 = c/M_1$$

Since $\lambda_7 = 0$ along the final orbit, it follows that $\lambda = 1$ at the final point of the IT arc. Simultaneous solution of system (2.8) subject to the condition $f = \theta - \omega$ yields

$$\begin{aligned}
&f_1 = 0.8736,\quad B_1 = 0.4022,\quad \varphi_1 = 0.4022,\quad D_1 = 0.5300 \\
&f_2 = 0.6625,\quad B_2 = 0.4099,\quad \varphi_2 = 0.2211,\quad D_2 = 0.5538
\end{aligned}$$

The time of motion over the IT arc amounts to 436.57 s. The ratio of the required characteristic velocity to the corresponding angular velocity is 1.4749, whereas the ratio or the impulse case is 1.2964. The results of solving this problem show that, as far as fuel consumption is concerned, the requirements of impulse-thrust and intermediate-thrust flight are comparable.

However, analysis of the system of equations (2.7) suggests that IT flight with a sufficiently large C value is preferable to the flight considered here.

REFERENCES

1. LAWDEN D. F., *Optimum Trajectories for Space Navigation*. Plenum Press, New York, 1964.
2. AZIZOV A. G. and KORSHUNOVA N. A., The problem of determining the trajectories in a Newtonian field. In *Problems of the Mechanics of Controlled Motion*. Perm. Gos. Univ., Perm', pp. 5–10, 1984.
3. ROBBINS H. M., Optimality of intermediate-thrust arcs of rocket trajectories. *AIAA J.* **3**, 6, 1094–1098, 1965.

4. ZAVALISHCHIN S. T., A supplement to Lawden's theory. *Prikl. Mat. Mekh.* **53**, 5, 731–738. 1989.
5. AZIZOV A. G. and KORSHUNOVA N. A., The use of the Levi-Civita method to analyse optimum trajectories. *Kosmich. Issledovaniya* **17**, 3, 378–386. 1979.
6. AZIMOV D. M., Spherical trajectories for intermediate-thrust arcs and their application. In *Proceedings of the Scientific Conference on "Modelling of Complex Mechanical Systems"*, Tashkent, 1991.
7. KORSHUNOVA N. A., Optimum rocket trajectories in a central field. In *Mechanics and Applied Mathematics.* Trudy Tashkent. Gos. Univ., No. 476a, pp. 85–94, 1975.
8. McCUE G. A., Quasilinearization determination of optimum finite-thrust orbital transfers. *AIAA J.* **5**, 4, 755–763, 1967.
9. EHRICKE K., *Space Flight*, Vol. 2. Nauka, Moscow, 1969.

Translated by D.L.